

Due Fri

1.5 – Elementary Matrices and a Method for Finding A^{-1}

Definition 1: Matrices A and B are said to be **row equivalent** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

Useful for developing theory

Definition 2: A matrix E is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

Theorem 1.5.1 Row Operations by Matrix Multiplication

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A .

6. An elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A .

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

a) $E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

b) $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

a) $R_1 \rightarrow -6R_1$

$R_1 \rightarrow -6R_1$

$$EA = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{bmatrix}$$

b) $\underline{R_2} \rightarrow \underline{R_2} - 4R_1$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$$

$\rightarrow R_2 \rightarrow R_2 - 4R_1$

$$= \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ -7 & 1 & -1 & 21 & 19 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$$

| | | | | | |
|--|----|----|----|----|----|
| | 1 | -3 | -1 | 5 | 3 |
| | -8 | 4 | 0 | 16 | 16 |
| | -7 | 1 | -1 | 21 | 19 |

\leftarrow Same \rightarrow

Pf Thm 1.5.1, Case I: E results from multiplying the i^{th} row of I by k .

Let $\vec{e}_1 = [1 \ 0 \ 0 \ \dots \ 0]$, $\vec{e}_2 = [0 \ 1 \ 0 \ \dots \ 0]$,
 $\vec{e}_m = [0 \ 0 \ 0 \ \dots \ 1]$, $k\vec{e}_i = [0 \ 0 \ \dots \ k \ \dots \ 0]$

Recall from 1.3 that

$$EA = \begin{bmatrix} \vec{e}_1 A \\ \vec{e}_2 A \\ \vdots \\ \underline{k\vec{e}_i A} \\ \vdots \\ \vec{e}_m A \end{bmatrix}$$

which results in multiplying the i^{th} row of A by k .

7. Use the following matrices and find an elementary matrix E that satisfies the stated equation.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix},$$

$$D = \begin{bmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{bmatrix}, F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

a. $EA = B$

b. $EB = A$

c. $EA = C$ Row operation: $R_3 \rightarrow R_3 - 2R_1$

d. $EC = A$

Then $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Check: $EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix} = C$ ✓

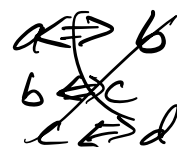
Theorem 1.5.2 Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Pf is based on: Row operations are reversible.

Theorem 1.5.3 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all are true or all false.

- a) A is invertible. — one and not more than one
- b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- c) The reduced row echelon form of A is I_n .
- d) A is expressible as a product of elementary matrices.



Pf: $a \Rightarrow b$ A invertible $\Rightarrow A^{-1}$ exists.

Then $A^{-1}A = I$.

$$\begin{aligned} \text{So } A\vec{x} = \vec{0} &\Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{0} \\ &\Rightarrow \vec{x} = \vec{0} \end{aligned}$$

Suppose \vec{x}_1 is also a solution to $A\vec{x} = \vec{0}$.

$$\text{Then } A\vec{x}_1 = \vec{0} \Rightarrow A\vec{x} - A\vec{x}_1 = \vec{0}$$

$$\begin{aligned} A(\vec{x} - \vec{x}_1) = \vec{0} &\Rightarrow A^{-1}A(\vec{x} - \vec{x}_1) = A^{-1}\vec{0} \\ &\Rightarrow \vec{x} - \vec{x}_1 = \vec{0} \Rightarrow \vec{x} = \vec{x}_1. \quad \checkmark \end{aligned}$$

$b \Rightarrow c$ $A\vec{x} = \vec{b}$ has only the trivial solution,
so the system has no free variables.

By Thm 1.2.1 rref has n nonzero rows.

By Thm 1.4.3, rref is I_n . ✓

$c \Rightarrow d$ rref of A is I_n , so I_n can
be obtained from A by a series of
elementary row operations. We can express

$$I_n = E_r \cdots E_2 E_1 A \quad \text{by Thm 1.5.1}$$

Then $E_1^{-1} E_2^{-1} \cdots E_r^{-1} I = A$, a product
of elementary matrices.

$d \Rightarrow a$ the matrix product $E_r \cdots E_2 E_1$
in the previous step is A^{-1}
 $\Rightarrow A$ is invertible. ✓

Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

$$15. \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

A^{-1}

based on the proof of $c \Rightarrow d$

$$\rightarrow [A | I_n] \rightarrow [I_n | A^{-1}]$$

Since

$$I_n = E_r \cdots E_2 E_1 A, \text{ we have}$$

$$A^{-1} = \bar{E}_r \cdots \bar{E}_2 \bar{E}_1 I_n$$

So the elementary row operations that reduce A to I_n will yield A^{-1} when applied to I_n .

$$\left(\begin{array}{ccc|ccc} \cancel{2} & \cancel{6} & \cancel{6} & 1 & 0 & 0 \\ \cancel{2} & \cancel{7} & \cancel{6} & 0 & 1 & 0 \\ \cancel{2} & \cancel{7} & \cancel{7} & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \quad \begin{array}{l} 2 \ 7 \ 6 \ 0 \ 1 \ 0 \\ -2 \ -6 \ -6 \ -1 \ 0 \ 0 \\ \hline 0 \ 1 \ 0 \ -1 \ 1 \ 0 \end{array} \quad \begin{array}{l} 2 \ 7 \ 7 \ 0 \ 0 \ 1 \\ -2 \ -6 \ -6 \ -1 \ 0 \ 0 \\ \hline 0 \ 1 \ 1 \ -1 \ 0 \ 1 \end{array}$$

last

$$\left(\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right) R_3 \rightarrow R_3 - R_2 \quad \begin{array}{l} 0 \ 1 \ 1 \ -1 \ 0 \ 1 \\ 0 \ -1 \ 0 \ 1 \ -1 \ 0 \\ \hline 0 \ 0 \ 1 \ 0 \ -1 \ 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right) R_1 \rightarrow R_1 - 6R_3 \quad \begin{array}{l} 2 \ 6 \ 6 \ 1 \ 0 \ 0 \\ 0 \ 0 \ -6 \ 0 \ 6 \ -6 \\ \hline 2 \ 6 \ 0 \ 1 \ 6 \ -6 \end{array} \quad \begin{array}{l} 2 \ 6 \ 0 \ 1 \ 6 \ -6 \\ 0 \ 1 \ 0 \ -1 \ 1 \ 0 \\ 0 \ 0 \ 1 \ 0 \ -1 \ 1 \end{array}$$

$$R_1 \rightarrow R_1 - 6R_2$$
$$\begin{array}{cccccc} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & -6 & 0 & 6 & -6 & 0 \\ \hline 2 & 0 & 0 & 7 & 0 & -6 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

$$R_1 \rightarrow \frac{1}{2}R_1$$

$$A^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

18.
$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$$

① ② ③ last

Check later:

$$A^{-1} = \begin{bmatrix} -4/5 & 3/5 & 1/5 & 1/5 \\ 3/2 & 0 & -1 & 0 \\ 1/2 & 0 & 0 & 0 \\ 4/5 & 2/5 & -1/5 & -1/5 \end{bmatrix}$$

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array} \right)$$

$R_1 \leftrightarrow R_2$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array} \right)$$

19. Find the inverse of each of the following 4×4 matrices, where k_1, k_2, k_3, k_4, k are all nonzero.

a.
$$\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$$

b.
$$\begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a.
$$\begin{bmatrix} 1/k_1 & 0 & 0 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 0 & 0 & 1/k_4 \end{bmatrix}$$

b.
$$\left[\begin{array}{cccc|cccc} k & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{bmatrix} 1/k & -1/k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/k & -1/k \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

27. Show that the matrices A and B are row equivalent by finding a sequence of elementary row operations that produces B from A , and then use that result to find a matrix C such that $CA = B$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
 Corresponding
Elementary matrices

To obtain B from A :

$$R_2 \rightarrow R_2 - R_1$$

$$\begin{array}{r} 1 \ 4 \ 1 \\ -1 \ 2 \ -3 \\ \hline 0 \ 2 \ -2 \end{array} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - R_2$$

$$\begin{array}{r} 1 \ 2 \ 3 \\ 0 \ -2 \ 2 \\ \hline 1 \ 0 \ 5 \end{array} \quad \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_1$$

$$\begin{array}{r} 2 \ 1 \ 9 \\ -1 \ 0 \ -5 \\ \hline 1 \ 1 \ 4 \end{array} \quad \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = B$$

$$C = E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

24. Express the matrix and its inverse as products of elementary matrices.

$$\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$$

$$E_r \cdots E_2 E_1 A = I$$

$$\text{So } A^{-1} = E_r \cdots E_2 E_1$$

$$\text{and } A = E_1^{-1} E_2^{-1} \cdots E_r^{-1}$$

Reduce A to I_2 , keeping track of the elementary matrices

$$\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix} \quad R_2 \rightarrow R_2 + 5R_1$$

$$\begin{array}{cc} -5 & 2 \\ \hline 5 & 0 \\ \hline 0 & 2 \end{array}$$

$$E_1 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad R_2 \rightarrow \frac{1}{2} R_2$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^{-1} = E_2 E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$A = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

check:

$$= \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix} = A \checkmark$$

33. Prove that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A .

Let E_1, E_2, \dots, E_r be elementary matrices such that when the associated row operations are applied to A , the result is B .

That is, $E_r \cdots E_2 E_1 A = B$

By Thm 1.5.2, each E_i is invertible.

$$\text{so } A = E_1^{-1} E_2^{-1} \cdots E_r^{-1} B$$

Since E_i^{-1} is an elementary matrix, there is a sequence of elementary row operations that when applied to B gives A .